Transformations of random regressions

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Various polynomials can be used in random regression models. Theoretically, they result in equivalent models. In practice, some polynomials lead to easier computing and fewer numerical problems.

Let the value of an i-th level of a random effect for point t be:

$$u_i(t) = \sum_{j=0}^{n} \alpha_j(t)\, u_{ij} = \alpha(t)'\, u_i, \quad var(u_i)=G_0, \quad var[u_i(t)] = \alpha(t)'\, G_0\, \alpha(t)$$

where u_i is the vector of random effect of animal i, and α_j is the j-th covariable.

One can redefine the covariables by:

$$\alpha(t)'\, P\, P^{-1}\, u_i = \alpha^*(t)'\, u_i^*, \quad var(u_i^*)=G_0^*$$

where

$$\alpha^*(t)' = \alpha(t)'\, P \quad \text{and} \quad u_i^* = P^{-1}\, u_i$$

To obtain covariables, random effects and covariances on the original scale:

$$\alpha(t)' = \alpha^*(t)'\, P^{-1}, \qquad u_i = P\, u_i^*$$

Covariances and on the original and transformed scales are:

$$var(u_i^*) = G_0^* = var(P^{-1}\, u_i) = P^{-1} G_0\, P^{-1'}$$

$$G_0 = P\, G_0^*\, P'$$

In general, one can have a series of transformations:

$$P = P_1\, P_2\, ..\, P_n$$

One special transformation results in diagonal matrix of covariances. Let
G_0=VDV'
where V are eigenvectors and D is a diagonal matrix of eigenvalues. Then:

$$\alpha^*(t)' = \alpha(t)'\, V \quad \text{and} \quad var(u_i^*) = V'\, G_0\, V = V'\, VDV'\, V = D$$

Example

Let $\alpha(t)'=[1\ t\ t^2]$, $t \in$,0,1>.

The transformation to a scale <-1,1> would be:

$$\alpha^*(t)' = [1 \ (2t-1) \ (2t-1)^2] = [1 \ t \ t^2] \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

The transformation matrix from standardized regular polynomials to Legendre polynomials is:

$$P_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{5}{8}} \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{45}{8}} \end{bmatrix} = \begin{bmatrix} 0.7071 & 0 & -0.7906 \\ 0 & 1.2247 & 0 \\ 0 & 0 & 2.3717 \end{bmatrix}$$

Thus, transformation from original polynomials to Legendre polynomials is $P_s P_1$

$$P_S \ P_1 = \begin{bmatrix} 0.7071 & -1.2247 & 1.5811 \\ 0 & 2.4495 & -9.4868 \\ 0 & 0 & 9.4868 \end{bmatrix}$$

The covariables on the transformed scale are:
$[.7071 \quad -1.2247+2.4495t \quad 1.5811-9.4868t+9.4868t^2]$

If G_0 estimated with the original polynomials is:

$$G_0 = \begin{bmatrix} 20 & -30 & 20 \\ -30 & 80 & -70 \\ 20 & -70 & 90 \end{bmatrix}$$

with the transformed covariables it will be estimated close to:

$$P_1^{-1} \ P_s^{-1} \ \ G_0 \ \ P_s^{-1'} \ P_1^{-1'} = \begin{bmatrix} 20.0 & 0.962 & 2.236 \\ 0.962 & 5.0 & 0.861 \\ 2.236 & 0.861 & 1.0 \end{bmatrix}$$

This matrix by chance is diagonal. In general, the diagonal matrices can be obtained only by using eigenvectors.

$$G_0 = VDV', \quad V = = \begin{bmatrix} 0.7554 & -0.6106 & 0.2378 \\ 0.5820 & 0.4585 & -0.6716 \\ 0.3011 & 0.6457 & 0.7018 \end{bmatrix}, \quad D = \begin{bmatrix} 4.8558 & 0 & 0 \\ 0 & 21.377900 & \\ 0 & 0 & 163.7663 \end{bmatrix}$$

If P=V, then

$$P_d^{-1}\ G_0\ P_d^{-1'} = D'$$

$$\alpha(t)'\ V = [0.755 + .582t + .301t^2 \quad -.610 + .458*t + .645*t^2 \quad .237 - .671*t + .701*t^2]$$

Please note that 2.5% variance is associated with the first eigenvalue, 11% with the second, and 86% with the last. Each eigenvalue is associated with a specific eigenvector. For example, eigenvector associated with the largest eigenvalue defines a function $.237-.671*t+.701t^2$, which could be interpreted as the average shape of the trajectory.