Transformations of random regressions

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Various polynomials can be used in random regression models. Theoretically, they result in equivalent models. In practice, some polynomials lead to easier computing and fewer numerical problems.

Let the value of an i-th level of a random effect for point t be:

\[ u_i(t) = \sum_{j=0}^{n} \alpha_j(t) u_{ij} \]

where \( u \) is the vector of random effect of animal i, and \( \alpha_j \) is the j-th covariable.

One can redefine the covariables by:

\[ \alpha^*(t)' = \alpha(t)' P \quad \text{and} \quad u_i^* = P^{-1} u_i \]

To obtain covariables, random effects and covariances on the original scale:

\[ \alpha^*(t)' = \alpha^*(t)' P^{-1}, \quad u_i = P u_i^* \]

Covariances and on the original and transformed scales are:

\[ \text{var}(u_i^*) = G_0^* = \text{var}(P^{-1} u_i) = P^{-1}G_0 P^{-1}' \]

In general, one can have a series of transformations:

\[ P = P_1 P_2 \ldots P_n \]

One special transformation results in diagonal matrix of covariances. Let

\[ G_0 = VDV' \]

where V are eigenvectors and D is a diagonal matrix of eigenvalues. Then:

\[ \alpha^*(t)' = \alpha(t)' V \quad \text{and} \quad \text{var}(u_i^*) = V' G_0 V = V' VDV' V = D \]

Example

Let \( \alpha(t)' = [1 \ t \ t^2], \ t \in [0,1] \).

The transformation to a scale \([-1,1]\) would be:
The transformation matrix from standardized regular polynomials to Legendre polynomials is:

\[
\begin{bmatrix}
1 & -1 & 1 \\
0 & 2 & -4 \\
0 & 0 & 4
\end{bmatrix}
\]

Thus, transformation from original polynomials to Legendre polynomials is:

\[
P = \begin{bmatrix}
0.7071 & 0 & -0.7906 \\
0 & 1.2247 & 0 \\
0 & 0 & 2.3717
\end{bmatrix}
\]

The transformation matrix from standardized regular polynomials to Legendre polynomials is:

\[
P_1 = \begin{bmatrix}
\frac{1}{2} & 0 & -\frac{\sqrt{5}}{8} \\
0 & \frac{3}{2} & 0 \\
0 & 0 & \frac{45}{8}
\end{bmatrix}
\]

Thus, transformation from original polynomials to Legendre polynomials is:

\[
P_0 = P_1 \begin{bmatrix}
0.7071 & -1.2247 & 1.5811 \\
0 & 2.4495 & -9.4868 \\
0 & 0 & 9.4868
\end{bmatrix}
\]

The covariables on the transformed scale are:

\[
\begin{bmatrix}
0.7071 & -1.2247 + 2.4495t & 1.5811 - 9.4868t + 9.4868t^2
\end{bmatrix}
\]

If \( G_0 \) estimated with the original polynomials is:

\[
G_0 = \begin{bmatrix}
20 & -30 & 20 \\
-30 & 80 & -70 \\
20 & -70 & 90
\end{bmatrix}
\]

with the transformed covariables it will be estimated close to:

\[
P_0^{-1} G_0 P_1^{-1} \begin{bmatrix}
20.0 & 0.962 & 2.236 \\
0.962 & 5.0 & 0.861 \\
2.236 & 0.861 & 1.0
\end{bmatrix}
\]

This matrix by chance is diagonal. In general, the diagonal matrices can be obtained only by using eigenvectors.

\[
G_0 = \mathbf{V}\mathbf{D}\mathbf{V}'
\]

\[
\mathbf{V} = \begin{bmatrix}
0.7554 & -0.6106 & 0.2378 \\
0.5820 & 0.4585 & -0.6716 \\
0.3011 & 0.6457 & 0.7018
\end{bmatrix}
\]

\[
\mathbf{D} = \begin{bmatrix}
4.8558 & 0 & 0 \\
0 & 21.377900 & 0 \\
0 & 0 & 163.7663
\end{bmatrix}
\]
If $P=V$, then

$$P_d^{-1} G_b P_d^{-1} = D'$$

$$\omega(t)' V = [0.755 + 0.582t + 0.301t^2, -0.610 + 0.458t + 0.645t^2, 0.237 - 0.671t + 0.701t^2]$$

Please note that 2.5% variance is associated with the first eigenvalue, 11% with the second, and 86% with the last. Each eigenvalue is associated with a specific eigenvector. For example, eigenvector associated with the largest eigenvalue defines a function $0.237 - 0.671t + 0.701t^2$, which could be interpreted as the average shape of the trajectory.